Large time behavior of solutions to a semilinear hyperbolic system with relaxation(Mathematical Analysis in Fluid and Gas Dynamics)

数理解析研究所講究録
1536 巻 2007 年 151-171

Large time behavior of solutions to a semilinear hyperbolic system with relaxation

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Abstract

We are concerned with the initial value problem for a damped wave equation with a nonlinear convection term which is derived from a semilinear hyperbolic system with relaxation. We show the global existence and asymptotic decay of solutions in $W^{1,p}$ $(1 \leq p \leq \infty)$ under smallness condition on the initial data. Moreover, we show that the solution approaches in $W^{1,p}$ $(1 \leq p \leq \infty)$ the nonlinear diffusion wave expressed in terms of the self-similar solution of the Burgers equation as time tends to infinity. Our results are based on the detailed pointwise estimates for the fundamental solutions to the linearlized equation.

1 Introduction

We consider a nonlinear relaxation system of the form:

$$u_t + v_x = 0, \qquad v_t + u_x = f(u) - v, \tag{1.1}$$

where u and v are unknown functions of $t > 0$ and $x \in \mathbb{R}$, and $f(u)$ is a smooth function of u under consideration. If we eliminate v from (1.1), we obtain the following damped wave equation with a nonlinear convection term:

$$u_{tt} - u_{xx} + u_t + f(u)_x = 0. \tag{1.2}$$

We consider the initial value problem for (1.2) with the initial conditions

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x). \tag{1.3}$$

This initial value problem was studied by R. Orive and E. Zuazua [5] when $f(u) = |u|^{\gamma-1}u$ with $\gamma \geq 2$, so that $f'(0) = 0$. They proved the global existence and asymptotic decay of solutions under smallness condition on the initial data u_0 in $H^1 \cap L^1$ and u_1 in $L^2 \cap L^1$. Moreover, under the additional condition u_0, $u_1 \in L^2_1$, they observed that when $\gamma = 2$, the solution obtained approaches the self-similar solution $z(t,x)$ of the Burgers equation $z_t + (|z|z)_x = z_{xx}$ which verifies the integral condition $\int z(t,x)dx = M$, where $M := \int (u_0 + u_1)(x)dx$. When $\gamma > 2$, it was also observed in [5] that the asymptotic profile $z(t,x)$ is given by the heat kernel, i.e., the self-similar solution of the heat equation $z_t = z_{xx}$ which verifies $\int z(t,x)dx = M$ with the same M.

The main purpose in this paper is to generalize the results in [5] to the case where $f(u)$ satisfies the so called sub-characteristic condition $|f'(0)| < 1$. In addition, we develop L^p theory for p including $p = 1$. In fact, under smallness condition on the initial data u_0 in $W^{1,p} \cap L^1$ and u_1 in $L^p \cap L^1$, where $1 \le p \le \infty$, we prove that the solution exists globally in time and satisfies the decay estimates

$$\begin{aligned} &\|u(t)\|_{L^q} \le C(1+t)^{-\frac{1}{2}(1-\frac{1}{q})} \quad \text{for any } 1 \le q \le \infty, \\ &\|\partial_x u(t)\|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}. \end{aligned} \tag{1.4}$$

To discuss more detailed large-time behavior of the solution for $|f'(0)| < 1$, we need additional consideration. To see this, as in [3,1], we apply the Chapman-Enskog expansion to (1.1) and derive a viscous conservation law

$$w_t + f(w)_x = (\mu(w)w_x)_x \tag{1.5}$$

as the second order approximation of the expansion, where $\mu(w) = 1 - (f'(w))^2$. Note that the sub-characteristic condition $|f'(w)| < 1$ implies the parabolicity of (1.5). It is expected that the solution of (1.2) can be approximated by the solution of (1.5) or its simpler version

$$w_t + \left(\alpha w + \frac{\beta}{2}w^2\right)_x = \mu w_{xx}, \tag{1.6}$$

where $\alpha = f'(0)$, $\beta = f''(0)$ and $\mu = 1 - (f'(0))^2$. When $\beta = f''(0) > 0$, by the change of independent and dependent variables $x = y + \alpha t$ and $w = \beta z$, (1.6) is reduced to the Burgers equation $z_t + (z^2/2)_y = \mu z_{yy}$ whose asymptotic profile is given by its self-similar solution (see (2.3) below). Consequently, it is expected that the solution $u(t,x)$ of (1.2) is approximated by the nonlinear diffusion wave $w(t,x)$ which is a modification of the self-similar solution $z(t,y)$ of the Burgers equation and is defined as $w(t,x) = \beta^{-1}z(t, x-\alpha t)$. In fact, under the additional condition u_0, $u_1 \in L^1_1$, we show that the solution to the problem (1.2), (1.3) approaches the nonlinear diffusion wave $w(t,x)$ which verifies the integral condition $\int w(t,x)dx = M$, where $M := \int (u_0 + u_1)(x)dx$. More specifically, we show that

$$\begin{aligned} &\|(u-w)(t)\|_{L^q} \le C(1+t)^{-\frac{1}{2}(1-\frac{1}{q})-\frac{1}{2}+\varepsilon} \quad \text{for any } 1 \le q \le \infty, \\ &\|\partial_x (u-w)(t)\|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-1+\varepsilon} \end{aligned} \tag{1.7}$$

as $t \to \infty$, where ε is any fixed positive number.

Before closing this section, we give some notations used in this paper. Let $\mathcal{F}[f]$ denote the Fourier transform and $\mathcal{F}^{-1}[f]$ denote the Fourier inverse transform of f defined by

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) := \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx, \qquad \mathcal{F}^{-1}[f](x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)e^{ix\xi}d\xi.$$

For $1 \leq p \leq \infty$, we denote by $L^p = L^p(\mathbb{R})$ the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$. Let k be a nonnegative integer. Then $W^{k,p} = W^{k,p}(\mathbb{R})$ denotes the Sobolev space of L^p functions, equipped with the norm $\|f\|_{W^{k,p}}$. For $\alpha \in \mathbb{R}$, let $L^p_\alpha = L^p_\alpha(\mathbb{R})$ denote the weighted L^p space with the norm $\|f\|_{L^p_\alpha} := \|(1+|x|)^\alpha f\|_{L^p}$. Let X be a Banach space and let I be an interval on $\mathbb{R}$. Then $C(I;X)$ denotes the space of continuous functions on the interval I with values in the Banach space X. Also, $L^\infty(I;X)$ denotes the space of L^∞ functions on I with values in X.

2 Main results

In this section we give statements of our main results in this paper. The first result is concerning the global existence and optimal decay of solutions to the initial value problem (1.2), (1.3), which can be stated as follows.

Theorem 2.1 *Suppose that $|f'(0)| < 1$. Let $1 \leq p \leq \infty$ and assume that $u_0 \in W^{1,p} \cap L^1$ and $u_1 \in L^p \cap L^1$. Put*

$$E_0 := \|u_0\|_{W^{1,p}} + \|u_0\|_{L^1} + \|u_1\|_{L^p} + \|u_1\|_{L^1}.$$

Then there is a positive constant δ_0 such that if $E_0 \leq \delta_0$, then the initial value problem (1.2), (1.3) has a unique global solution $u(t,x)$ with

$$u \in C([0,\infty);\, W^{1,p} \cap L^1).$$

Moreover, the solution satisfies

$$\begin{aligned} \|u(t)\|_{L^q} &\leq CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{q})}, \\ \|\partial_x u(t)\|_{L^p} &\leq CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}, \end{aligned} \tag{2.1}$$

for any q with $1 \leq q \leq \infty$ and C is a constant.

Remark 2.2 *When $p = \infty$, we should replace the solution space by $C([0,\infty);\, L^1) \cap L^\infty((0,\infty);\, W^{1,\infty})$.*

In order to state our second main result concerning the large-time behavior of the solution obtained in Theorem 2.1, we define the nonlinear diffusion wave for (1.2). Consider the self-similar solution to the Burgers equation

$$z_t + (z^2/2)_x = \mu z_{xx}, \tag{2.2}$$

where $\mu = 1 - (f'(0))^2$, which is a solution of the form $z(t,x) = t^{-\frac{1}{2}}\phi(\frac{x}{\sqrt{t}})$. We denote by $z(t,x) = Z(t,x;\, \mu, M)$ the self-similar solution which satisfies the integral condition $\int z(t,x)dx = M$, where M is a parameter. This self-similar solution is given explicitly as

$$Z(t,x;\, \mu, M) = \sqrt{\frac{\mu}{t}}\, \frac{(e^{\frac{M}{2\mu}}-1)e^{-y^2}}{\sqrt{\pi} + (e^{\frac{M}{2\mu}}-1)\displaystyle\int_y^\infty e^{-\xi^2}d\xi}, \qquad y = \frac{x}{\sqrt{4\mu t}}. \tag{2.3}$$

We then define $W(t,x)$ by

$$W(t,x) = \beta^{-1} Z(t, x - \alpha t;\ \mu, \beta M), \tag{2.4}$$

where $\alpha = f'(0)$, $\beta = f''(0)$ and $\mu = 1 - (f'(0))^2$. Here we assumed that $\beta = f''(0) > 0$. We see that this $W(t,x)$ has the conserved quantity $\int W(t,x)dx = M$ and satisfies (1.6), i.e.,

$$w_t + \left(\alpha w + \frac{\beta}{2} w^2\right)_x = \mu w_{xx}, \tag{2.5}$$

which is an approximation to the viscous conservation law (1.5) derived form (1.1) by applying the Chapman-Enskog expansion. We call $W(t,x)$ defined by (2.4) the *nonlinear diffusion wave* for (1.2) if the parameter M is chosen as $M = \int (u_0 + u_1)(x)dx$.

The nonlinear diffusion wave defined above gives the large-time description of the solution obtained in Theorem 2.1.

Theorem 2.3 *Suppose that $|f'(0)| < 1$ and $f''(0) > 0$. Let $1 \le p \le \infty$ and assume that $u_0 \in W^{1,p} \cap L^1_1$ and $u_1 \in L^p \cap L^1_1$. Let $u(t,x)$ be the global solution of the problem (1.2), (1.3) constructed in Theorem 2.1, and let $W(t,x)$ be the nonlinear diffusion wave defined by (2.4) with $M = \int (u_0 + u_1)(x)dx$. Put $w(t,x) = W(t+1,x)$ and*

$$E_1 := \|u_0\|_{W^{1,p}} + \|u_0\|_{L^1_1} + \|u_1\|_{L^p} + \|u_1\|_{L^1_1}.$$

Then, for any ε with $0 < \varepsilon < \frac{1}{2}$, there is a positive constant δ_1 such that if $E_0 \le \delta_1$ (where E_0 is given in Theorem 2.1), then we have the following asymptotic relations:

$$\begin{aligned} \|(u-w)(t)\|_{L^q} &\le CE_1(1+t)^{-\frac{1}{2}(1-\frac{1}{q})-\frac{1}{2}+\varepsilon}, \\ \|\partial_x(u-w)(t)\|_{L^p} &\le CE_1(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-1+\varepsilon}, \end{aligned} \tag{2.6}$$

for any q with $1 \le q \le \infty$ and C is a constant.

Remark 2.4 *A straightforward computation using (2.4) and (2.3) yields*

$$\|\partial_x^l w(t)\|_{L^q} \le C|M|(1+t)^{-\frac{1}{2}(1-\frac{1}{q})-\frac{l}{2}} \tag{2.7}$$

for any $1 \le q \le \infty$ and $l = 0, 1, \cdots$, where $M = \int (u_0 + u_1)(x)dx$. More precisely, when $M \ne 0$, $\partial_x^l w(t,x)$ behaves exactly like $t^{-\frac{1}{2}(1-\frac{1}{q})-\frac{l}{2}}$ in L^q as $t \to \infty$. Therefore, the estimate (2.6) gives meaningful asymptotic relations for $t \to \infty$, provided that $M \ne 0$.

3 Fundamental solutions

The aim of this section is to study the fundamental solutions to the linearized equation of (1.2):

$$u_{tt} - u_{xx} + u_t + \alpha u_x = 0, \tag{3.1}$$

where $\alpha = f'(0)$. To this end, we consider (3.1) with the initial data

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x). \tag{3.2}$$

We take the Fourier transform, obtaining

$$\hat{u}_{tt} + \hat{u}_t + (\xi^2 + \alpha i\xi)\hat{u} = 0, \qquad \hat{u}(0,\xi) = \hat{u}_0(\xi), \quad \hat{u}_t(0,\xi) = \hat{u}_1(\xi). \tag{3.3}$$

The characteristic equation of (3.3) is $\lambda^2 + \lambda + (\xi^2 + \alpha i\xi) = 0$ and the eigenvalues are

$$\lambda_1(\xi) = \frac{1}{2}\Big(-1 + \sqrt{1 - 4(\xi^2 + \alpha i\xi)}\Big), \quad \lambda_2(\xi) = \frac{1}{2}\Big(-1 - \sqrt{1 - 4(\xi^2 + \alpha i\xi)}\Big). \tag{3.4}$$

The problem (3.3) is then solved as

$$\hat{u}(t,\xi) = \hat{G}(t,\xi)(\hat{u}_0(\xi) + \hat{u}_1(\xi)) + \hat{H}(t,\xi)\hat{u}_0(\xi), \tag{3.5}$$

where

$$\begin{aligned} \hat{G}(t,\xi) &= \frac{1}{\lambda_1(\xi) - \lambda_2(\xi)}\big(e^{\lambda_1(\xi)t} - e^{\lambda_2(\xi)t}\big), \\ \hat{H}(t,\xi) &= \frac{1}{\lambda_1(\xi) - \lambda_2(\xi)}\Big((1 + \lambda_1(\xi))e^{\lambda_2(\xi)t} - (1 + \lambda_2(\xi))e^{\lambda_1(\xi)t}\Big). \end{aligned} \tag{3.6}$$

We take the Fourier inverse transform of (3.5). This yields the solution formula of the linearized problem (3.1), (3.2):

$$u(t) = G(t) * (u_0 + u_1) + H(t) * u_0, \tag{3.7}$$

where $G(t,x)$ and $H(t,x)$ denote the Fourier inverse transforms of $\hat{G}(t,\xi)$ and $\hat{H}(t,\xi)$ in (3.6), respectively:

$$G(t,x) := \mathcal{F}^{-1}[\hat{G}(t,\cdot)](x), \quad H(t,x) := \mathcal{F}^{-1}[\hat{H}(t,\cdot)](x), \tag{3.8}$$

and $*$ denotes the convolution with respect to x. We call $G(t,x)$ and $H(t,x)$ the *fundamental solutions* of linearized damped wave equation (3.1).

We are interested in the asymptotic expressions of the fundamental solutions together with their detailed pointwise estimates. To state the results, we introduce the modified heat kernel:

$$G_0(t,x) = \frac{1}{\sqrt{4\pi\mu t}}e^{-(x-\alpha t)^2/4\mu t}, \tag{3.9}$$

where $\alpha = f'(0)$ and $\mu = 1 - (f'(0))^2$, which is the fundamental solution to the linear heat equation $w_t + \alpha w_x = \mu w_{xx}$. Then the result for $G(t,x)$ can be stated as follows.

Theorem 3.1 *Let $\alpha = f'(0)$ and $\mu = 1 - (f'(0))^2$, and assume that $|\alpha| < 1$. For each nonnegative integer l, the fundamental solution $G(t,x)$ can be expressed as*

$$G(t,x) = G_0(t,x) + G_\infty^{(l)}(t,x) + R^{(l)}(t,x) = G_\infty^{(l)}(t,x) + R_\infty^{(l)}(t,x).$$

Here $G_0(t,x)$ is the modified heat kernel in (3.9), and $G_\infty^{(l)}(t,x)$ is the singular part given as follows: We have $G_\infty^{(0)}(t,x) \equiv 0$ and

$$\partial_x^l G_\infty^{(l)}(t,x) = \sum_{k=0}^{l-1}\big\{e^{-\kappa t}P_k(t)\partial_x^{l-k-1}\delta(x+t) + e^{-\nu t}Q_k(t)\partial_x^{l-k-1}\delta(x-t)\big\} \tag{3.10}$$

for $l \geq 1$, where $\kappa = (1+\alpha)/2$, $\nu = (1-\alpha)/2$, $P_k(t)$ and $Q_k(t)$ are some polynomials of t of degree k, and δ denotes the Dirac delta function. The remainder terms $R^{(l)}(t,x)$ and $R^{(l)}_\infty(t,x)$ verify the following pointwise estimates:

$$
\begin{aligned}
|\partial_x^l R^{(l)}(t,x)| &\leq Ct^{-\frac{l+1}{2}}(1+t)^{-\frac{1}{2}}e^{-c(x-\alpha t)^2/t} + Ce^{-c(t+|x|)}, \\
|\partial_x^l R^{(l)}_\infty(t,x)| &\leq C(1+t)^{-\frac{l+1}{2}}e^{-c(x-\alpha t)^2/t} + Ce^{-c(t+|x|)}
\end{aligned}
\tag{3.11}
$$

for $l \geq 0$, where C and c are positive constants.

This theorem shows that the fundamental solution $G(t,x)$ can be well approximated by the modified heat kernel $G_0(t,x)$ as $t \to \infty$.

We have a similar expression also for $H(t,x)$.

Theorem 3.2 *Assume the same condition as in Theorem 3.1. For each $l \geq 0$, we can express $H(t,x)$ as*

$$H(t,x) = H^{(l)}_\infty(t,x) + S^{(l)}_\infty(t,x).$$

Here the singular part $H^{(l)}_\infty(t,x)$ is given as

$$\partial_x^l H^{(l)}_\infty(t,x) = \sum_{k=0}^{l}\{e^{-\kappa t}\tilde{P}_k(t)\partial_x^{l-k}\delta(x+t) + e^{-\nu t}\tilde{Q}_k(t)\partial_x^{l-k}\delta(x-t)\} \tag{3.12}$$

for $l \geq 0$, where κ and μ are the same as in Theorem 3.1, $\tilde{P}_k(t)$ and $\tilde{Q}_k(t)$ are some polynomials of t of degree k, and δ denotes the Dirac delta function. The remainder term satisfies the following pointwise estimate:

$$|\partial_x^l S^{(l)}_\infty(t,x)| \leq C(1+t)^{-\frac{l+2}{2}}e^{-c(x-\alpha t)^2/t} + Ce^{-c(t+|x|)} \tag{3.13}$$

for $l \geq 0$, where C and c are positive constants.

As a corollary of the above pointwise estimates of the fundamental solutions, we have the following L^p–L^q estimates for solutions to the linearized equation (3.1).

Corollary 3.3 *Assume the same condition as in Theorem 3.1 and let $1 \leq q \leq p \leq \infty$. Then we have the following L^p–L^q estimates:*

$$
\begin{aligned}
\|G(t)*\phi\|_{L^p} &\leq C(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}\|\phi\|_{L^q}, \\
\|\partial_x^l G(t)*\phi\|_{L^p} &\leq C(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{l}{2}}\|\phi\|_{L^q} + Ce^{-ct}\|\phi\|_{W^{l-1,p}}, \quad l \geq 1,
\end{aligned}
\tag{3.14}
$$

and

$$\|\partial_x^l H(t)*\phi\|_{L^p} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{l+1}{2}}\|\phi\|_{L^q} + Ce^{-ct}\|\phi\|_{W^{l,p}}, \quad l \geq 0. \tag{3.15}$$

Moreover, the solution operator $G(t)*$ is approximated by $G_0(t)*$ in the following sense:

$$
\begin{aligned}
\|(G-G_0)(t)*\phi\|_{L^p} &\leq Ct^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}(1+t)^{-\frac{1}{2}}\|\phi\|_{L^q}, \\
\|\partial_x^l(G-G_0)(t)*\phi\|_{L^p} &\leq Ct^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{l}{2}}(1+t)^{-\frac{1}{2}}\|\phi\|_{L^q} + Ce^{-ct}\|\phi\|_{W^{l-1,p}}, \quad l \geq 1.
\end{aligned}
\tag{3.16}
$$

Here C and c are some positive constants.

We can prove Corollary 3.3 by using Theorem 3.1, 3.2, and omit here.

4 Fundamental solution in Fourier space

In this section, under the condition $|f'(0)| < 1$, we consider $\hat{G}(t,\xi)$ and $\hat{H}(t,\xi)$ in (3.6) and derive their pointwise estimates, which are crucial in the proof of Theorems 3.1 and 3.2. Here ξ is regarded as a complex variable, i.e., $\xi \in \mathbb{C}$. We divide our computations into three parts corresponding to the low frequency region $|\xi| \le r_0$, the middle frequency region $r_0 \le |\xi| \le K_0$ and the high frequency region $|\xi| \ge K_0$, respectively. We omit the proof in this section.

In the low frequency region we have:

Lemma 4.1 *There is a positive constant r_0 such that for any $\xi \in \mathbb{C}$ with $|\xi| \le r_0$, we have the following expressions:*

$$\begin{aligned} &\hat{G}(t,\xi) = \hat{G}_0(t,\xi) + \hat{R}_0(t,\xi), \\ &\hat{G}_0(t,\xi) = e^{-(\alpha i\xi+\mu\xi^2)t}, \quad \hat{R}_0(t,\xi) = e^{-(\alpha i\xi+\mu\xi^2)t}\hat{R}_{0,1}(t,\xi) + e^{-t}\hat{R}_{0,2}(t,\xi) \end{aligned} \tag{4.1}$$

and

$$\hat{H}(t,\xi) = e^{-(\alpha i\xi+\mu\xi^2)t}\hat{H}_1(t,\xi) + e^{-t}\hat{H}_2(t,\xi). \tag{4.2}$$

Here $\alpha = f'(0)$, $\mu = 1-(f'(0))^2$, and

$$\begin{aligned} &|\hat{R}_{0,1}(t,\xi)| \le C|\xi|(1+|\xi|^2 t)e^{C|\xi|^3 t}, \quad |\hat{R}_{0,2}(t,\xi)| \le Ce^{C|\xi|t}, \\ &|\hat{H}_1(t,\xi)| \le C|\xi|e^{C|\xi|^3 t}, \quad |\hat{H}_2(t,\xi)| \le Ce^{C|\xi|t} \end{aligned} \tag{4.3}$$

for $|\xi| \le r_0$, where C is a positive constant.

Remark 4.2 *For $\hat{G}(t,\xi)$, we have another expression:*

$$\hat{G}(t,\xi) = e^{-(\alpha i\xi+\mu\xi^2)t}\hat{G}_1(t,\xi) + e^{-t}\hat{G}_2(t,\xi) \tag{4.4}$$

with $\hat{G}_1(t,\xi) = 1 + \hat{R}_{0,1}(t,\xi)$ and $\hat{G}_2(t,\xi) = \hat{R}_{0,2}(t,\xi)$ satisfying

$$|\hat{G}_1(t,\xi)| \le Ce^{C|\xi|^3 t}, \quad |\hat{G}_2(t,\xi)| \le Ce^{C|\xi|t}. \tag{4.5}$$

Next we consider in the high frequency region.

Lemma 4.3 *For each nonnegative integer l, there is a positive constant K_0 such that for any $\xi \in \mathbb{C}$ with $|\xi| \ge K_0$, we have the following expressions:*

$$\hat{G}(t,\xi) = \hat{G}^{(l)}_\infty(t,\xi) + \hat{R}^{(l)}_\infty(t,\xi), \quad \hat{H}(t,\xi) = \hat{H}^{(l)}_\infty(t,\xi) + \hat{S}^{(l)}_\infty(t,\xi). \tag{4.6}$$

Here $\hat{G}^{(0)}_\infty(t,\xi) \equiv 0$,

$$\begin{aligned} &\hat{G}^{(l)}_\infty(t,\xi) = \sum_{k=0}^{l-1}\{e^{-(\kappa-i\xi)t}P_k(t) + e^{-(\nu+i\xi)t}Q_k(t)\}(i\xi)^{-k-1}, \quad l \ge 1, \\ &\hat{R}^{(l)}_\infty(t,\xi) = \{e^{-(\kappa-i\xi)t}P_l(t) + e^{-(\nu+i\xi)t}Q_l(t)\}(i\xi)^{-l-1} \\ &\qquad\qquad + e^{-(\kappa-i\xi)t}\hat{R}^{(l)}_{\infty,1}(t,\xi) + e^{-(\nu+i\xi)t}\hat{R}^{(l)}_{\infty,2}(t,\xi), \quad l \ge 0, \end{aligned} \tag{4.7}$$

and

$$
\begin{aligned}
\hat{H}_{\infty}^{(l)}(t,\xi) &= \sum_{k=0}^{l} \{e^{-(\kappa-i\xi)t}\tilde{P}_k(t) + e^{-(\nu+i\xi)t}\tilde{Q}_k(t)\}(i\xi)^{-k}, \quad l \geq 0, \\
\hat{S}_{\infty}^{(l)}(t,\xi) &= \{e^{-(\kappa-i\xi)t}\tilde{P}_l(t) + e^{-(\nu+i\xi)t}\tilde{Q}_l(t)\}(i\xi)^{-l-1} \\
&\qquad + e^{-(\kappa-i\xi)t}\hat{S}_{\infty,1}^{(l)}(t,\xi) + e^{-(\nu+i\xi)t}\hat{S}_{\infty,2}^{(l)}(t,\xi), \quad l \geq 0,
\end{aligned}
\tag{4.8}
$$

where $\kappa = (1+\alpha)/2$, $\nu = (1-\alpha)/2$ *with* $\alpha = f'(0)$, *and* $P_k(t)$, $Q_k(t)$, $\tilde{P}_k(t)$ *and* $\tilde{Q}_k(t)$ *are polynomials of* t *of degree* k. *Moreover, we have*

$$
\begin{aligned}
|\hat{R}_{\infty,1}^{(l)}(t,\xi)| + |\hat{R}_{\infty,2}^{(l)}(t,\xi)| &\leq C|\xi|^{-l-2}(1+t)^{l+1}e^{C|\xi|^{-1}t}, \\
|\hat{S}_{\infty,1}^{(l)}(t,\xi)| + |\hat{S}_{\infty,2}^{(l)}(t,\xi)| &\leq C|\xi|^{-l-2}(1+t)^{l+2}e^{C|\xi|^{-1}t}
\end{aligned}
\tag{4.9}
$$

for $|\xi| \geq K_0$, *where* C *is a positive constant.*

In the middle frequency region, as in [2], we derive the corresponding estimates by employing the energy method in the Fourier space.

Lemma 4.4 *We write* $\xi = \eta + i\zeta$, *where* $\eta, \zeta \in \mathbb{R}$. *Then, for any* $r > 0$, *there exists a positive constant* $\sigma(r)$ *depending on* r *such that if* $|\eta| \geq r$ *and* $|\zeta| \leq \sigma(r)$, *then we have the following estimates:*

$$
|\hat{G}(t,\xi)| \leq C(1+|\eta|)^{-1}e^{-c\rho(\eta)t}, \quad |\hat{H}(t,\xi)| \leq Ce^{-c\rho(\eta)t}, \tag{4.10}
$$

where $\rho(\eta) = \frac{\eta^2}{1+\eta^2}$, *and* C *and* c *are positive constants independent of* r.

5 Proof of pointwise estimates

In this section, following [4,2], we give the proof of Theorems 3.1 and 3.2 concerning the pointwise estimates of the fundamental solutions.

Proof of Theorem 3.1. For each nonnegative integer l, we express $\hat{G}$ in (3.6) as

$$
\hat{G}(t,\xi) = \hat{G}_0(t,\xi) + \hat{G}_{\infty}^{(l)}(t,\xi) + \hat{R}^{(l)}(t,\xi), \tag{5.1}
$$

where $\hat{G}_0$ and $\hat{G}_{\infty}^{(l)}$ are given explicitly in (4.1) and (4.7), respectively, and $\hat{R}^{(l)}$ is defined by (5.1). We write the Fourier inverse transform of (5.1) as

$$
G(t,x) = G_0(t,x) + G_{\infty}^{(l)}(t,x) + R^{(l)}(t,x). \tag{5.2}
$$

Here the first two terms on the right hand side of (5.2) can be given explicitly. In this proof, we consider the derivative $\partial_x^l R^{(l)}(t,x)$ and $\partial_x^l R_{\infty}^{(l)}(t,x)$ of the remainder terms.

Lemma 5.1 *For each* $l \geq 0$, *we have the following estimate:*

$$
|\partial_x^l R^{(l)}(t,x)| \leq C(1+t)^{-\frac{l+2}{2}}e^{-c(x-\alpha t)^2/t} + Ce^{-c(t+|x|)} \tag{5.3}
$$

for $t \geq 1$, *where* C *and* c *are positive constants.*

Proof. We have

$$\partial_x^l R^{(l)}(t,x) = \mathcal{F}^{-1}[(i\xi)^l \hat{R}^{(l)}(t,\cdot)](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi)^l \hat{R}^{(l)}(t,\xi)e^{i\xi x}d\xi$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi)^l \hat{R}^{(l)}(t,\xi)e^{i\xi x}d\eta \qquad (\xi = \eta + i\zeta),$$

where, thanks to the Cauchy integral theorem, we have changed the path of integration from the real axis to the straight line $\xi = \eta + i\zeta$ (with a small fixed ζ specified later) which is parallel to the real axis. We divide the above integral into three parts corresponding to the regions $|\eta| \le r$, $r \le |\eta| \le K$ and $|\eta| \ge K$, respectively, where $r > 0$ and $K > 0$ are constants which will be specified later. Now we recall the relations

$$\hat{R}^{(l)} = \hat{R}_0 - \hat{G}_\infty^{(l)}, \quad \hat{R}^{(l)} = \hat{G} - \hat{G}_0 - \hat{G}_\infty^{(l)}, \quad \hat{R}^{(l)} = \hat{R}_\infty^{(l)} - \hat{G}_0,$$

which follow from (4.1), (4.6) and (5.1). We then substitute these three relations into the above integral over the regions $|\eta| \le r$, $r \le |\eta| \le K$, and $|\eta| \ge K$, respectively. Consequently, we obtain

$$\begin{aligned} 2\pi\, \partial_x^l R^{(l)}(t,x) &= \int_{|\eta|\le r}(i\xi)^l \hat{R}_0 e^{i\xi x}d\eta + \int_{r\le|\eta|\le K}(i\xi)^l \hat{G}e^{i\xi x}d\eta \\ &+ \int_{|\eta|\ge K}(i\xi)^l \hat{R}_\infty^{(l)} e^{i\xi x}d\eta - \int_{|\eta|\le K}(i\xi)^l \hat{G}_\infty^{(l)} e^{i\xi x}d\eta - \int_{|\eta|\ge r}(i\xi)^l \hat{G}_0 e^{i\xi x}d\eta \\ &=: I_1 + I_2 + I_3 - I_4 - I_5. \end{aligned} \tag{5.4}$$

where $\xi = \eta + i\zeta$. We choose ζ according to the point (t,x) as follows:

$$\begin{aligned} &\zeta = \delta(x-\alpha t)/t \quad \text{if} \quad |x-\alpha t|/t \le 1, \\ &\zeta = \delta \quad \text{if} \quad |x-\alpha t|/t \ge 1 \quad \text{and } x - \alpha t > 0, \\ &\zeta = -\delta \quad \text{if} \quad |x-\alpha t|/t \ge 1 \quad \text{and } x - \alpha t < 0, \end{aligned} \tag{5.5}$$

where $\delta > 0$ is a small constant which will be specified later. Note that in any case we have $|\xi|^2 \le |\eta|^2 + \delta^2$. For the moment, we assume that r and δ are so small that $r^2 + \delta^2 \le r_0^2$ and $\delta \le \sigma(r)$, while K is so large that $K \ge K_0$, where r_0, K_0 and $\sigma(r)$ are the constants in Lemmas 4.1, 4.3 and 4.4, respectively.

<u>Case 1.</u> Consider the case where $|x - \alpha t|/t \le 1$. In this case we take $\zeta = \delta(x-\alpha t)/t$ by (5.5) so that $\xi = \eta + i\delta(x-\alpha t)/t$.

First, we rewrite the term I_1 by using (4.1) as

$$I_1 = \int_{|\eta|\le r} e^{-\mu\xi^2 t}(i\xi)^l \hat{R}_{0,1} e^{i\xi(x-\alpha t)}d\eta + e^{-t}\int_{|\eta|\le r}(i\xi)^l \hat{R}_{0,2}e^{i\xi x}d\eta =: I_{1,1} + I_{1,2}.$$

We substitute the first pointwise estimate in (4.3) into $I_{1,1}$ and then use the simple relations $-\mathrm{Re}(\mu\xi^2 t) = -\mu\eta^2 t + \mu\delta^2(x-\alpha t)^2/t$ and $\mathrm{Re}(i\xi(x-\alpha t)) = -\delta(x-\alpha t)^2/t$. This gives

$$|I_{1,1}| \le C\int_{|\eta|\le r}|e^{-\mu\xi^2 t}|\,|\xi|^{l+1}(1+|\xi|^2 t)e^{C|\xi|^3 t}|e^{i\xi(x-\alpha t)}|d\eta \le C(1+t)^{-\frac{l+2}{2}}e^{-\gamma(x-\alpha t)^2/t},$$

provided that δ and r are suitably small, where γ is a positive constant such that $\gamma < \delta$. Similarly, using the second pointwise estimate in (4.3) and the relation $\mathrm{Re}(i\xi x) \le -\delta(x-\alpha t)^2/t + |\alpha|\delta t$, we have

$$|I_{1,2}| \le Ce^{-t}\int_{|\eta|\le r} |\xi|^l e^{C|\xi|t}|e^{i\xi x}|d\eta \le Ce^{-ct}e^{-\delta(x-\alpha t)^2/t},$$

provided that δ and r are suitably small, where c is a positive constant with $c<1$. Here we have used the inequality $|\xi|^2 \le |\eta|^2+\delta^2$. Thus we have

$$|I_1| \le C(1+t)^{-\frac{l+2}{2}}e^{-\gamma(x-\alpha t)^2/t}. \tag{5.6}$$

Next we estimate I_2. When $r\le|\eta|\le K$, we have from (4.10) that $|\hat{G}| \le Ce^{-c_0r^2t}$, provided that r is suitably small and K is suiatbly large, where c_0 is a positive constant independent of r and K. Therefore, noting that $\mathrm{Re}(i\xi x) \le -\delta(x-\alpha t)^2/t + |\alpha|\delta t$, we obtain

$$|I_2| \le \int_{r\le|\eta|\le K} |\xi|^l|\hat{G}||e^{i\xi x}|d\eta \le Ce^{-ct}e^{-\delta(x-\alpha t)^2/t}, \tag{5.7}$$

provided that δ is suitably small depending on r, where c is a positive constant with $c<c_0r^2$.

For I_3, we use the expression of $\hat{R}^{(l)}_\infty$ in (4.7) and write I_3 as

$$\begin{aligned} I_3 &= e^{-\kappa t}\int_{|\eta|\ge K}\{P_l(t)(i\xi)^{-1}+(i\xi)^l\hat{R}^{(l)}_{\infty,1}\}e^{i\xi(x+t)}d\eta \\ &\quad + e^{-\nu t}\int_{|\eta|\ge K}\{Q_l(t)(i\xi)^{-1}+(i\xi)^l\hat{R}^{(l)}_{\infty,2}\}e^{i\xi(x-t)}d\eta =: I_3^+ + I_3^-. \end{aligned}$$

Moreover, we rewrite I_3^+ as

$$\begin{aligned} I_3^+ &= e^{-\kappa t}P_l(t)\int_{|\eta|\ge K}(i\eta)^{-1}e^{i\xi(x+t)}d\eta + e^{-\kappa t}P_l(t)\int_{|\eta|\ge K}((i\xi)^{-1}-(i\eta)^{-1})e^{i\xi(x+t)}d\eta \\ &\quad + e^{-\kappa t}\int_{|\eta|\ge K}(i\xi)^l\hat{R}^{(l)}_{\infty,1}e^{i\xi(x+t)}d\eta =: I_{3,1}^+ + I_{3,2}^+ + I_{3,3}^+. \end{aligned}$$

We estimate each term as follows. For $I_{3,1}^+$, we see that

$$I_{3,1}^+ = e^{-\kappa t}P_l(t)e^{-\delta(x-\alpha t)(x+t)/t}\int_{|\eta|\ge K}(i\eta)^{-1}e^{i\eta(x+t)}d\eta$$

because $i\xi(x+t) = -\delta(x-\alpha t)(x+t)/t + i\eta(x+t)$. Here we observe that $e^{-\delta(x-\alpha t)(x+t)/t} \le e^{-\delta(x-\alpha t)^2/t}e^{c_1\delta t}$ with $c_1 = 1+\alpha$, and that

$$\int_{|\eta|\ge K}(i\eta)^{-1}e^{i\eta(x+t)}d\eta = 2\int_K^\infty \frac{\sin\eta(x+t)}{\eta}d\eta = 2\,\mathrm{sign}(x+t)\int_{|x+t|K}^\infty \frac{\sin y}{y}dy,$$

which is uniformly bounded. Consequently, we obtain

$$|I_{3,1}^+| \le C(1+t)^l e^{-\kappa t}e^{-\delta(x-\alpha t)^2/t}e^{c_1\delta t} \le Ce^{-ct}e^{-\delta(x-\alpha t)^2/t},$$

provided that δ is suitably small, where c is a positive constant with $c<\kappa$. Also, for $I_{3,2}^{+}$, we have

$$(i\xi)^{-1}-(i\eta)^{-1}=\frac{1}{i\eta-\delta(x-\alpha t)/t}-\frac{1}{i\eta}=\frac{\delta(x-\alpha t)/t}{i\eta(i\eta-\delta(x-\alpha t)/t)}=O(|\eta|^{-2}),$$

and $|e^{i\xi(x+t)}|\leq e^{-\delta(x-\alpha t)^2/t}e^{c_1\delta t}$ with $c_1=1+\alpha$. Hence we obtain

$$|I_{3,2}^{+}|\leq C(1+t)^{l}e^{-\kappa t}\int_{|\eta|\geq K}|(i\xi)^{-1}-(i\eta)^{-1}||e^{i\xi(x+t)}|d\eta\leq Ce^{-ct}e^{-\delta(x-\alpha t)^2/t}$$

for suitably small δ, where $0<c<\kappa$. Similarly, making use of the pointwise estimate of $\hat{R}_{\infty,1}^{(l)}$ in (4.9), we have

$$|I_{3,3}^{+}|\leq C(1+t)^{l+1}e^{-\kappa t}\int_{|\eta|\geq K}|\xi|^{-2}e^{C|\xi|^{-1}t}|e^{i\xi(x+t)}|d\eta\leq Ce^{-ct}e^{-\delta(x-\alpha t)^2/t},$$

provided that δ is suitably small and K is suitably large, where $0<c<\kappa$. Summarizing all these computations, we have $|I_3^{+}|\leq Ce^{-ct}e^{-\delta(x-\alpha t)^2/t}$. Another term I_3^{-} can be estimated just in the same way. Thus we arrive at the estimate

$$|I_3|\leq Ce^{-ct}e^{-\delta(x-\alpha t)^2/t}. \tag{5.8}$$

The fourth term I_4 can be treated more easily. We have from (4.7) that

$$I_4=\sum_{k=0}^{l-1}\{e^{-\kappa t}P_k(t)\int_{|\eta|\leq K}(i\xi)^{l-k-1}e^{i\xi(x+t)}d\eta+e^{-\nu t}Q_k(t)\int_{|\eta|\leq K}(i\xi)^{l-k-1}e^{i\xi(x-t)}d\eta\}.$$

Here we note that $|e^{i\xi(x\pm t)}|\leq e^{-\delta(x-\alpha t)^2/t}e^{c_1\delta t}$ with $c_1=\max\{1+\alpha,1-\alpha\}$. Therefore, letting $\kappa_1=\min\{\kappa,\nu\}$, we have

$$|I_4|\leq C(1+t)^{l-1}e^{-\kappa_1 t}\int_{|\eta|\leq K}(1+|\xi|)^{l}(|e^{i\xi(x+t)}|+|e^{i\xi(x-t)}|)d\eta\leq Ce^{-ct}e^{-\delta(x-\alpha t)^2/t}. \tag{5.9}$$

for suitably small δ, where $0<c<\kappa_1$.

Finally, we estimate the term I_5 which is rewritten by using the expression of $\hat{G}_0$ in (4.1) as

$$I_5=\int_{|\eta|\geq r}(i\xi)^{l}e^{-\mu\xi^2 t}e^{i\xi(x-\alpha t)}d\eta.$$

We have

$$|I_5|\leq\int_{|\eta|\geq r}|\xi|^{l}|e^{-\mu\xi^2 t}||e^{i\xi(x-\alpha t)}|d\eta\leq Ct^{-\frac{l+1}{2}}e^{-ct}e^{-\gamma(x-\alpha t)^2/t}, \tag{5.10}$$

provided that δ is suitably small, where $0<\gamma<\delta$.

All these computations from (5.6) to (5.10) prove the desired estimate (5.3) for $|x-\alpha t|/t\leq 1$.

Case 2. Next we consider the case where $|x-\alpha t|/t\geq 1$ and $x-\alpha t>0$. (The case where $|x-\alpha t|/t\geq 1$ and $x-\alpha t<0$ can be treated just in the same way and we omit this final case.) In this case we take $\zeta=\delta$ by (5.5) so that $\xi=\eta+i\delta$.

For the term $I_{1,1}$, we see that $\mathrm{Re}(\xi^2) = \eta^2 - \delta^2$ and $\mathrm{Re}(i\xi(x-\alpha t)) = -\delta|x-\alpha t|$. Also, we find that $e^{-\delta|x-\alpha t|} \le e^{-\delta t/2}e^{-\delta|x-\alpha t|/2}$ because of $|x-\alpha t| \ge t$. Therefore we have

$$|I_{1,1}| \le C\int_{|\eta|\le r} |e^{-\mu\xi^2 t}|\,|\xi|^{l+1}(1+|\xi|^2 t)e^{C|\xi|^3 t}|e^{i\xi(x-\alpha t)}|d\eta \le Ce^{-\gamma_1 t}e^{-\delta|x-\alpha t|/2} \le Ce^{-\gamma(t+|x|)},$$

provided that δ and r are suitably small, where $0<\gamma<\gamma_1<\delta/2$.For the term $I_{1,2}$, noting that $\mathrm{Re}(i\xi x) = -\delta x \le -\delta|x-\alpha t| + |\alpha|\delta t$, we have

$$|I_{1,2}| \le Ce^{-t}\int_{|\eta|\le r} |\xi|^l e^{C|\xi|t}|e^{i\xi x}|d\eta \le Ce^{-ct}e^{-\delta|x-\alpha t|} \le Ce^{-\gamma(t+|x|)},$$

provided that δ and r are suitably small, where $0<c<1$ and $0<\gamma<\min\{c,\delta\}$. Thus we have

$$|I_1| \le Ce^{-\gamma(t+|x|)}. \tag{5.11}$$

Similarly, for the term I_2, I_3 and I_4, we can replace the factor $\delta|x-\alpha t|^2/t$ in (5.7), (5.8) and (5.9) by $\delta|x-\alpha t|$ and obtain

$$|I_2|,\ |I_3|,\ |I_4| \le Ce^{-ct}e^{-\delta|x-\alpha t|} \le Ce^{-\gamma(t+|x|)}, \tag{5.12}$$

provided that δ and r are suitably small and K is suitably large, where c is a certain positive constant and $0<\gamma<\min\{c,\delta\}$. Also, for the term I_5, we have

$$|I_5| \le \int_{|\eta|\ge r} |\xi|^l|e^{-\mu\xi^2 t}|\,|e^{i\xi(x-\alpha t)}|d\eta \le Ct^{-\frac{l+1}{2}}e^{-\gamma_1 t}e^{-\delta|x-\alpha t|/2} \le Ct^{-\frac{l+1}{2}}e^{-\gamma(t+|x|)}, \tag{5.13}$$

provided that δ is suitably small, where $0<\gamma<\gamma_1<\delta/2$. All these observations show the desired estimate (5.3) for $|x-\alpha t|/t \ge 1$ and hence the proof of Lemma 5.1 is complete. □

The pointwise estimate of $\partial_x^l R^{(l)}(t,x)$ given in Lemma 5.1 contains the additional singularity at $t=0$ (see the term I_5 in (5.13)). For the proof of Theorem 3.1 we must remove this singularity. To this end, we recall (4.6) and write

$$\hat{G}(t,\xi) = \hat{G}_\infty^{(l)}(t,\xi) + \hat{R}_\infty^{(l)}(t,\xi) \tag{5.14}$$

for each $l \ge 0$, where $\hat{G}_\infty^{(l)}(t,\xi)$ is given explicitly in (4.7). We write the Fourier inverse transform of (5.14) as

$$G(t,x) = G_\infty^{(l)}(t,x) + R_\infty^{(l)}(t,x), \tag{5.15}$$

where $\partial_x^l G_\infty^{(l)}(t,x)$ was given explicitly. We show that the remainder term $R_\infty^{(l)}(t,x)$ satisfies the pointwise estimate given in (3.11):

Lemma 5.2 *For each $l \ge 0$, we have the following pointwise estimate:*

$$|\partial_x^l R_\infty^{(l)}(t,x)| \le C(1+t)^{-\frac{l+1}{2}}e^{-c(x-\alpha t)^2/t} + Ce^{-c(t+|x|)} \tag{5.16}$$

for any $t>0$, where C and c are positive constants.

Proof. We have as the counterpart of (5.4) that

$$
\begin{aligned}
2\pi\partial_x^l R_\infty^{(l)}(t,x) &= \int_{-\infty}^{\infty}(i\xi)^l \hat{R}_\infty^{(l)}(t,\xi)d\eta \\
&= \int_{|\eta|\le r}(i\xi)^l \hat{G} e^{i\xi x} d\eta + \int_{r\le|\eta|\le K}(i\xi)^l \hat{G} e^{i\xi x} d\eta \\
&\quad + \int_{|\eta|\ge K}(i\xi)^l \hat{R}_\infty^{(l)} e^{i\xi x} d\eta - \int_{|\eta|\le K}(i\xi)^l \hat{G}_\infty^{(l)} e^{i\xi x} d\eta \\
&=: J_1 + J_2 + J_3 + J_4,
\end{aligned}
\tag{5.17}
$$

where $\xi = \eta + i\zeta$. Here we have used the relation $\hat{R}_\infty^{(l)} = \hat{G} - \hat{G}_\infty^{(l)}$ in the regions $|\eta| \le r$ and $r \le |\eta| \le K$. To estimate the term J_1, we compare it with I_1 in (5.4). In the present case, it suffices to use the expression (4.4) of $\hat{G}$ instead of the expression (4.1) of $\hat{R}_0$. This suggests that all the estimates for I_1 in the proof of Lemma 5.1 are valid also for J_1 if we replace the exponent $l+1$ appearing in the estimates for I_1 by l. In particular, as the counterpart of (5.6), we have

$$
|I_1| \le C(1+t)^{\frac{l+1}{2}} e^{-\gamma(x-\alpha t)^2/t}
$$

for $|x-\alpha t|/t \le 1$. The other terms in (5.17) are just the same as those in (5.4), namely, we have $J_2 = I_2$, $J_3 = I_3$ and $J_4 = I_4$. (Here we do not have any term like I_5 having the additional singularity at $t=0$.) These observations give the desired estimate (5.16). This complete the proof of Lemma 5.2. $\square$

Now, in order to complete the proof of Theorem 3.1, we show the estimate (3.11) for $\partial_x^l R^{(l)}(t,x)$. Namely, for each $l \ge 0$, we show that

$$
|\partial_x^l R^{(l)}(t,x)| \le Ct^{-\frac{l+1}{2}}(1+t)^{-\frac{1}{2}} e^{-c(x-\alpha t)^2/t} + Ce^{-c(t+|x|)}
\tag{5.18}
$$

for any $t > 0$. To see this, we recall the relation $R^{(l)} = R_\infty^{(l)} - G_0$ and estimate the right hand side of this equality. For the first term, we apply the estimate (5.16). For the second term, by a straightforward computation, we have $|\partial_x^l G_0(t,x)| \le Ct^{-\frac{l+1}{2}} e^{-c(x-\alpha t)^2/t}$. Thus we obtain

$$
|\partial_x^l R^{(l)}(t,x)| \le Ct^{-\frac{l+1}{2}} e^{-(x-\alpha t)^2/t} + Ce^{-c(t+|x|)}.
\tag{5.19}
$$

A combination of the estimates (5.3) for $t \ge 1$ and (5.19) for $0 < t \le 1$ yields the desired estimate (5.18). This completes the proof of Theorem 3.1. $\square$

The proof of Theorem 3.2 is similar to that of Lemma 5.2 and omitted here.

6 Global existence and decay

In this section we study the initial value problem (1.2), (1.3) and prove the global existence result stated in Theorem 2.1. First, we rewrite the equation (1.2) as

$$
u_{tt} - u_{xx} + u_t + \alpha u_x = -g(u)_x,
\tag{6.1}
$$

where $\alpha = f'(0)$ and $g(u) := f(u) - f(0) - f'(0)u = O(u^2)$. Then, applying the Duhamel principle, we transform the problem (1.2) (or (6.1)), (1.3) into the integral equation

$$u(t) = G(t) * (u_0 + u_1) + H(t) * u_0 - \int_0^t G(t-s) * g(u)_x(s)ds, \tag{6.2}$$

where $G(t,x)$ and $H(t,x)$ are the fundamental solutions to the linearized equation (3.1) and are defined in (3.8).

We want to solve the above integral equation by applying the contraction mapping principle. For this purpose, we define the mapping $\Phi[u]$ by

$$\Phi[u](t) := G(t) * (u_0 + u_1) + H(t) * u_0 - \int_0^t G(t-s) * g(u)_x(s)ds \tag{6.3}$$

and put

$$\Phi_0(t) := G(t) * (u_0 + u_1) + H(t) * u_0. \tag{6.4}$$

Let us consider in the Banach space X defined as follows: For $1 \le p < \infty$,

$$\begin{aligned} X &:= \{u \in C([0,\infty); W^{1,p} \cap L^1);\ \|u\|_X < \infty\}, \\ \|u\|_X &:= \sup_{t\ge0} \|u(t)\|_{L^1} + \sup_{t\ge0} (1+t)^{\frac{1}{2}(1-\frac{1}{p})+\frac{1}{2}} \|\partial_x u(t)\|_{L^p}. \end{aligned} \tag{6.5}$$

and for $p = \infty$,

$$\begin{aligned} X &:= \{u \in C([0,\infty); L^1) \cap L^\infty((0,\infty); W^{1,\infty});\ \|u\|_X \le \infty\}, \\ \|u\|_X &:= \sup_{t\ge0} \|u(t)\|_{L^1} + \sup_{t\ge0} (1+t) \|\partial_x u(t)\|_{L^\infty}. \end{aligned} \tag{6.6}$$

It is also useful to introduce

$$\|u\|_Y := \sup_{t\ge0} \|u(t)\|_{L^1} + \sup_{t\ge0} (1+t)^{\frac{1}{2}} \|u(t)\|_{L^\infty}. \tag{6.7}$$

Notice that

$$\|u(t)\|_{L^q} \le \|u\|_Y (1+t)^{-\frac{1}{2}(1-\frac{1}{q})} \tag{6.8}$$

for each q with $1 \le q \le \infty$, which follows from the inequality $\|u\|_{L^q} \le \|u\|_{L^\infty}^{1-1/q} \|u\|_{L^1}^{1/q}$ and the definition of $\|u\|_Y$. Also, we see that $\|u\|_Y \le C_* \|u\|_X$, where $C_* \ge 1$ is the constant appearing in the Gagliardo-Nirenberg inequality $\|u\|_{L^\infty} \le C_* \|\partial_x u\|_{L^p}^{\theta} \|u\|_{L^1}^{1-\theta}$ with $\theta = 1/(2 - 1/p)$.

Let us introduce a closed convex subset S_R of X by

$$S_R := \{u \in X;\ \|u\|_X \le R\}, \tag{6.9}$$

where $R > 0$ is a parameter which will be determined later. We wish to show that for a suitably chosen R, Φ becomes a contraction mapping of S_R. To this end, we prepare the following:

Lemma 6.1 *(i) Let $1 \le p \le \infty$ and assume that $u_0 \in W^{1,p} \cap L^1$ and $u_1 \in L^p \cap L^1$. Then we have*

$$\|\Phi_0\|_X \le C_0 E_0 \tag{6.10}$$

for some positive constant C_0, where E_0 is given in Theorem 2.1.
(ii) Let u, $v \in X$. For any given positive number M, we suppose that $\|u(t)\|_{L^\infty}$, $\|v(t)\|_{L^\infty} \le M$ for $t \ge 0$. Then we have

$$\|\Phi[u] - \Phi[v]\|_X \le C_1(\|u\|_X + \|v\|_X)\|u - v\|_X, \tag{6.11}$$

where $C_1 = C_1(M)$ is a positive constant depending on M.

Proof. We obtain the proof of (i) by using Corollary 3.3, and omit here. Let us show (ii). It follows from (6.3) that

$$\Phi[u](t) - \Phi[v](t) = -\int_0^t \partial_x G(t-s) * \big(g(u) - g(v)\big)(s)ds. \tag{6.12}$$

Here we claim that

$$\begin{aligned} &\|g(u) - g(v)\|_{L^q} \le C(\|u\|_{L^\infty} + \|v\|_{L^\infty})\|u - v\|_{L^q}, \\ &\|\partial_x(g(u) - g(v))\|_{L^p} \le C\{(\|u\|_{L^\infty} + \|v\|_{L^\infty})\|\partial_x(u - v)\|_{L^p} \\ &\qquad\qquad + (\|\partial_x u\|_{L^p} + \|\partial_x v\|_{L^p})\|u - v\|_{L^\infty}\}, \end{aligned} \tag{6.13}$$

provided that $\|u\|_{L^\infty}$, $\|v\|_{L^\infty} \le M$, where $1 \le p,\, q \le \infty$, and $C = C(M)$ denotes a constant depending on M. This follows from the fact that $g(u) = O(u^2)$ and hence $g(u) - g(v) = a(u,v)(u - v)$ with a function $a(u,v) = O(|u| + |v|)$. Consequently, we have in terms of $\|\cdot\|_X$ and $\|\cdot\|_Y$ that

$$\begin{aligned} &\|\big(g(u) - g(v)\big)(t)\|_{L^q} \le C(\|u\|_Y + \|v\|_Y)\|u - v\|_Y(1+t)^{-\frac{1}{2}(1-\frac{1}{q})-\frac{1}{2}}, \\ &\|\partial_x\big(g(u) - g(v)\big)(t)\|_{L^p} \le C(\|u\|_X + \|v\|_X)\|u - v\|_X(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-1}, \end{aligned} \tag{6.14}$$

where $C = C(M)$. Now, we take the L^1 norm of (6.12) and apply (3.14) with $l = 1$, $p = q = 1$. Then, using the first estimate in (6.14), we have

$$\begin{aligned} \|\big(\Phi[u] - \Phi[v]\big)(t)\|_{L^1} &\le \int_0^t \|\partial_x G(t-s) * \big(g(u) - g(v)\big)(s)\|_{L^1} ds \\ &\le C(M)|[u,v]|_Y \int_0^t (1+t-s)^{-\frac{1}{2}}(1+s)^{-\frac{1}{2}} ds \le C(M)|[u,v]|_Y, \end{aligned} \tag{6.15}$$

where we wrote $|[u,v]|_Y := (\|u\|_Y + \|v\|_Y)\|u - v\|_Y$. Next, we want to estimate the derivative of (6.12). To this end, we decompose the integral on the right hand side of (6.12) into two pars and write $\Phi[u] - \Phi[v] = \Psi_1 + \Psi_2$, where Ψ_1 and Ψ_2 are corresponding to the integrations over $[0, t/2]$ and $[t/2, t]$, respectively. For the term $\partial_x \Psi_1$, we apply (3.14) with

$l = 2$, $q = 1$ and then make use of (6.14). Then, writing $|[u,v]|_X := (\|u\|_X + \|v\|_X)\|u-v\|_X$, we obtain

$$\begin{aligned}\|\partial_x \Psi_1(t)\|_{L^p} &\le \int_0^{t/2} \|\partial_x^2 G(t-s) * \big(g(u) - g(v)\big)(s)\|_{L^p} ds \\ &\le C(M)|[u,v]|_X (1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}},\end{aligned}$$

Similarly, for the term $\partial_x I_2$, we apply (3.14) with $l = 1$, $q = p$ and then use (6.14). This yields

$$\begin{aligned}\|\partial_x \Psi_2(t)\|_{L^p} &\le \int_{t/2}^{t} \|\partial_x G(t-s) * \partial_x\big(g(u) - g(v)\big)(s)\|_{L^p} ds \\ &\le C(M)|[u,v]|_X (1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}.\end{aligned}$$

Thus we have shown that

$$\|\partial_x\big(\Phi[u] - \Phi[v]\big)(t)\|_{L^p} \le C(M)|[u,v]|_X (1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}. \tag{6.16}$$

The desired estimate (6.11) follows from (6.15) and (6.16), and hence the proof of Lemma 6.1 is complete. □

Proof of Theorem 2.1. We determine the parameter R by $R := 2C_0E_0$, where C_0 is the positive constant in (6.10). For this choice of R, we suppose that u, $v \in S_R$. Then, we have $\|u\|_X \le R$ and hence $\|u\|_Y \le C_*\|u\|_X \le C_*R$ (the same for v), where $C_* \ge 1$ is the constant appeared in the previous Gagliardo-Nirenberg inequality. Therefore, we have from (6.11) that

$$\|\Phi[u] - \Phi[v]\|_X \le C_1(\|u\|_X + \|v\|_X)\|u-v\|_X \le 2C_1R\|u-v\|_X = 4C_0C_2(E_0)E_0\|u-v\|_X,$$

where the constant $C_1 = C_1(M)$ in (6.11) is evaluated at $M = C_*R = 2C_*C_0E_0$ and is denoted by $C_2(E_0)$. Consequently, we have

$$\|\Phi[u] - \Phi[v]\|_X \le \frac{1}{2}\|u-v\|_X, \tag{6.17}$$

provided that E_0 is so small that $4C_0C_2(E_0)E_0 \le \frac{1}{2}$. On the other hand, letting $v = 0$ in (6.17), we have

$$\|\Phi[u] - \Phi[0]\|_X \le R/2.$$

Therefore, noting that $\Phi[0] = \Phi_0$ and using (6.10), we obtain

$$\|\Phi[u]\|_X \le \|\Phi_0\|_X + \|\Phi[u] - \Phi[0]\|_X \le C_0E_0 + R/2 = R \tag{6.18}$$

Thus we have shown by (6.17) and (6.18) that Φ is a contraction mapping of S_R, provided that $4C_0C_2(E_0)E_0 \le \frac{1}{2}$. Hence we can conclude that the mapping Φ admits a unique fixed point u in S_R, namely, we have $u = \Phi[u]$. This fixed point u verifies the estimate (2.1) and is the desired global solution to the problem (1.2), (1.3). Thus the proof of Theorem 2.1 is complete. □

7 Asymptotic behavior

The aim of this section is to prove Theorem 2.3 concerning the asymptotic profile of the solution to the problem (1.2), (1.3).

We denote by $W(t,x)$ be the nonlinear diffusion wave defined by (2.4) with $M = \int (u_0+u_1)(x)dx$ and put $w(t,x) = W(t+1,x)$. Then this $w(t,x)$ solves (2.5) and hence the integral equation

$$w(t) = G_0(t) * w_0 - \frac{\beta}{2}\int_0^t G_0(t-s) * (w^2)_x(s)ds. \tag{7.1}$$

Here $G_0(t,x)$ is the fundamental solution to the linearized equation of (2.5) and is given by (3.9), and $w_0(x) := W(1,x)$ is a rapidly decreasing function satisfying $\int w_0(x)dx = M = \int (u_0+u_1)(x)dx$ and

$$\|w_0\|_{W^{1,p}} + \|w_0\|_{L^1_1} \le C|M| \le C\|u_0+u_1\|_{L^1} \le CE_0. \tag{7.2}$$

Let $u(t,x)$ be the global solution to the problem (1.2), (1.3) which was constructed in Theorem 2.1 as a solution to the integral equation (6.2). In order to study the difference $u(t,x) - w(t,x)$, we subtract (7.1) from (6.2), obtaining

$$\begin{aligned}(u-w)(t) &= (G-G_0)(t) * (u_0+u_1) + G_0(t) * (u_0+u_1-w_0)\\ &+ H(t) * u_0 - \int_0^t G(t-s) * \left(g(u) - \beta u^2/2\right)_x(s)ds\\ &- \frac{\beta}{2}\int_0^t (G-G_0)(t-s) * (u^2)_x(s)ds - \frac{\beta}{2}\int_0^t G_0(t-s) * (u^2-w^2)_x(s)ds\\ &=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6.\end{aligned} \tag{7.3}$$

We want to estimate the right hand side of (7.2). To do that, we need the following L^p-L^q estimate for the solution operator $G_0(t)*$.

Lemma 7.1 *([2]) Let $1 \le q \le p \le \infty$, and let $l \ge 0$ be an integer. Then we have*

$$\|\partial_x^l G_0(t) * \phi\|_{L^p} \le Ct^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{l}{2}}\|\phi\|_{L^q}. \tag{7.4}$$

Also, if $\int \phi(x)dx = 0$, then we have

$$\|\partial_x^l G_0(t) * \phi\|_{L^p} \le Ct^{-\frac{1}{2}(1-\frac{1}{p})-\frac{l}{2}}(1+t)^{-\frac{1}{2}}\|\phi\|_{L^1_1}. \tag{7.5}$$

Here C and c are positive constants.

The proof is given in Iguchi, Kawashima [2], and is omitted here.

Now we estimate (7.3) by introducing the following quantities:

$$M(t) := \sup_{0\le s\le t}(1+s)^{\frac{1}{2}-\varepsilon}\|(u-w)(s)\|_{L^1}, \quad N(t) := \sup_{0\le s\le t}(1+s)^{\frac{1}{2}(1-\frac{1}{p})+1-\varepsilon}\|\partial_x(u-w)(s)\|_{L^p}, \tag{7.6}$$

where ε is any fixed constant such that $0 < \varepsilon < \frac{1}{2}$.

Proof of Theorem 2.3. The proof consists of three claims below. First, we show the following L^1 estimate:

Claim 7.2 *There is a positive constant $\delta_1(\varepsilon)$ depending on ε such that if $E_0 \le \delta_1(\varepsilon)$, then we have*

$$\|(u-v)(t)\|_{L^1} \le CE_1(1+t)^{-\frac{1}{2}+\varepsilon}. \tag{7.7}$$

It suffices to estimate each term on the right hand side of (7.3). For the term I_1, we have from (3.16) that

$$\|I_1\|_{L^1} \le C(1+t)^{-\frac{1}{2}}\|u_0+u_1\|_{L^1} \le CE_0(1+t)^{-\frac{1}{2}}.$$

Also, since $\int(u_0+u_1-w_0)(x)dx = 0$, we have from (7.5) that

$$\|I_2\|_{L^1} \le C(1+t)^{-\frac{1}{2}}\|u_0+u_1-w_0\|_{L^1_1} \le CE_1(1+t)^{-\frac{1}{2}},$$

where we used (7.2). For I_3, we apply (3.15) to obtain

$$\|I_3\|_{L^1} \le C(1+t)^{-\frac{1}{2}}\|u_0\|_{L^1} \le CE_0(1+t)^{-\frac{1}{2}}.$$

Next, we estimate I_4 by applying (3.14) with $l=1$, $p=q=1$ as

$$\|I_4\|_{L^1} \le \int_0^t \|\partial_x G(t-s) * (g(u)-\beta u^2/2)(s)\|_{L^1} ds \le CE_0^3(1+t)^{-\frac{1}{2}}\log(2+t),$$

where we have used the fact that $g(u)-\beta u^2/2 = O(|u|^3)$ and the estimate (2.1). The term I_5 can be estimated similarly. In fact, we have from (3.16) with $l=1$, $p=q=1$ that

$$\|I_5\|_{L^1} \le C\int_0^t \|\partial_x(G-G_0)(t-s) * (u^2)(s)\|_{L^1} ds \le CE_0^2(1+t)^{-\frac{1}{2}}\log(2+t),$$

where we used (2.1). (A more delicate computation can give the present estimate without the factor $\log(2+t)$ but we omit it.) Finally, we estimate I_6 by applying (7.4) with $l=1$, $p=q=1$. We obtain

$$\|I_6\|_{L^1} \le C\int_0^t \|\partial_x G_0(t-s) * (u^2-w^2)(s)\|_{L^1} ds \le C(\varepsilon)E_0 M(t)(1+t)^{-\frac{1}{2}+\varepsilon}$$

for some constant $C(\varepsilon)$ depending on ε. Here we have used the inequality $\|u^2-w^2\|_{L^1} \le \|u+w\|_{L^\infty}\|u-w\|_{L^1}$ together with the estimates (2.1) and (2.7) and the definition of $M(t)$ in (7.6). Summarizing all these estimates, we arrive at

$$\|(u-w)(t)\|_{L^1} \le CE_1(1+t)^{-\frac{1}{2}} + CE_0^2(1+t)^{-\frac{1}{2}}\log(2+t) + C(\varepsilon)E_0M(t)(1+t)^{-\frac{1}{2}+\varepsilon}. \tag{7.8}$$

Since $\log(2+t) \le C(\varepsilon)(1+t)^\varepsilon$, this yields the inequality $M(t) \le CE_1 + C(\varepsilon)E_0^2 + C(\varepsilon)E_0M(t)$, from which follows the desired estimate $M(t) \le CE_1$ if E_0 is so small that $C(\varepsilon)E_0 \le \frac{1}{2}$. Thus we have shown the L^1 estimate (7.7).

Second, we derive the following L^∞ estimate:

Claim 7.3 *We have*

$$\|(u-v)(t)\|_{L^\infty} \le CE_1(1+t)^{-1+\varepsilon}, \tag{7.9}$$

provided that $E_0 \le \delta_2(\varepsilon)$ with a suitably small $\delta_2(\varepsilon)$.

For the term I_1, we apply (3.16) with $p=\infty$, $q=1$ and obtain

$$\|I_1\|_{L^\infty} \le Ct^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}}\|u_0+u_1\|_{L^1} \le CE_0t^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}}.$$

Also, for I_2, we apply (7.5) to obtain

$$\|I_2\|_{L^\infty} \le Ct^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}}\|u_0+u_1-w_0\|_{L^1_1} \le CE_1t^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}}.$$

Similarly, applying (3.15) with $p=\infty$, $q=1$, we have

$$\|I_3\|_{L^\infty} \le C(1+t)^{-1}\|u_0\|_{L^1} + Ce^{-ct}\|u_0\|_{L^\infty} \le CE_0(1+t)^{-1}.$$

Next, we estimate I_4 by applying (3.14) with $l=1$, $p=\infty$, $q=1$ as

$$\|I_4\|_{L^\infty} \le \int_0^t \|\partial_x G(t-s) * \left(g(u)-\beta u^2/2\right)(s)\|_{L^\infty} ds \le CE_0^3(1+t)^{-1}\log(2+t),$$

where we used (2.1). Similarly, we estimate I_5 by applying (3.16) with $l=0$, $p=\infty$, $q=1$. We obtain

$$\|I_5\|_{L^\infty} \le C\int_0^t \|(G-G_0)(t-s) * \partial_x(u^2)(s)\|_{L^\infty} ds \le CE_0^2(1+t)^{-1}\log(2+t),$$

where we have used the inequality $\|\partial_x(u^2)\|_{L^1} \le C\|u\|_{L^r}\|\partial_x u\|_{L^p}$ with $\frac{1}{p}+\frac{1}{r}=1$ and the estimate (2.1). Finally, we estimate I_6. We apply (7.4) with $l=1$, $p=\infty$, $q=1$ and then with $l=1$, $p=q=\infty$. A combination of the resulting two estimates gives

$$\|I_6\|_{L^\infty} \le C\int_0^t \|\partial_x G_0(t-s) * (u^2-w^2)(s)\|_{L^\infty} ds \le C(\varepsilon)E_0E_1(1+t)^{-1+\varepsilon}$$

for some constant $C(\varepsilon)$ depending on ε. Here we have used the inequalities $\|u^2-w^2\|_{L^1} \le \|u+w\|_{L^\infty}\|u-w\|_{L^1}$ and $\|u^2-w^2\|_{L^\infty} \le \|u\|_{L^\infty}^2 + \|w\|_{L^\infty}^2$ and the estimates (2.1), (2.7) and (7.7). Since $\log(2+t) \le C(\varepsilon)(1+t)^\varepsilon$, these observations show that

$$\|(u-w)(t)\|_{L^\infty} \le CE_1t^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}} + C(\varepsilon)E_0E_1(1+t)^{-1+\varepsilon}. \tag{7.10}$$

Therefore, assuming that $C(\varepsilon)E_0 \le 1$, we obtain

$$\|(u-w)(t)\|_{L^\infty} \le CE_1t^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}+\varepsilon}$$

This combined with (2.1) and (2.7) gives the desired estimate (7.9).

It remains to prove the following estimate for the derivative:

Claim 7.4 *We have*

$$\|\partial_x(u-v)(t)\|_{L^p} \le CE_1(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-1+\varepsilon}, \tag{7.11}$$

provided that $E_0 \le \delta_3(\varepsilon)$ with a suitably small $\delta_3(\varepsilon)$.

In the following we put $\gamma = \frac{1}{2}(1-\frac{1}{p})$. Notice that $0 \le \gamma \le \frac{1}{2}$ for $1 \le p \le \infty$. For the term $\partial_x I_1$, we apply (3.16) with $l=1$, $q=1$ and then with $l=1$, $q=p$, and combine them to obtain

$$\|\partial_x I_1\|_{L^p} \le Ct^{-\frac{1}{2}}(1+t)^{-\gamma-\frac{1}{2}}\big(\|u_0+u_1\|_{L^1} + \|u_0+u_1\|_{L^p}\big) \le CE_0 t^{-\frac{1}{2}}(1+t)^{-\gamma-\frac{1}{2}}.$$

Also, for $\partial_x I_2$, we apply (7.5) with $l=1$ and then (7.4) with $l=1$, $q=p$. A combination of the resulting two estimates gives

$$\|\partial_x I_2\|_{L^p} \le Ct^{-\frac{1}{2}}(1+t)^{-\gamma-\frac{1}{2}}\big(\|u_0+u_1-w_0\|_{L^1_1} + \|u_0+u_1-w_0\|_{L^p}\big) \le CE_1 t^{-\frac{1}{2}}(1+t)^{-\gamma-\frac{1}{2}}.$$

Similarly, applying (3.15) with $l=1$, $q=1$, we have

$$\|\partial_x I_3\|_{L^p} \le C(1+t)^{-\gamma-1}\|u_0\|_{L^1} + Ce^{-ct}\|u_0\|_{W^{1,p}} \le CE_0(1+t)^{-\gamma-1}.$$

Next, we we want to estimate the derivatives $\partial_x I_j$, $j=4,5,6$. To this end, we decompose each integral I_j into two parts and write $I_j = I_{j,1} + I_{j,2}$, where $I_{j,1}$ and $I_{j,2}$ are corresponding to the integrations over $[0,t/2]$ and $[t/2,t]$, respectively. Now, for the term $\partial_x I_{4,1}$, we apply (3.14) with $l=2$, $q=1$, obtaining

$$\|\partial_x I_{4,1}\|_{L^p} \le \int_0^{t/2} \|\partial_x^2 G(t-s) * \big(g(u)-\beta u^2/2\big)(s)\|_{L^p} ds \le CE_0^3(1+t)^{-\gamma-1}\log(2+t),$$

where we have used the estimates $\|\big(g(u)-\beta u^2/2\big)(s)\|_{L^1} \le CE_0^3(1+s)^{-1}$ and $\|\partial_x^l\big(g(u)-\beta u^2/2\big)(s)\|_{L^p} \le CE_0^3(1+s)^{-\gamma-1-\frac{l}{2}}$ $(l=0,1)$ which follow from (2.1). Also, applying (3.14) with $l=1$, $q=p$, we have

$$\|\partial_x I_{4,2}\|_{L^p} \le \int_{t/2}^{t} \|\partial_x G(t-s) * \partial_x\big(g(u)-\beta u^2/2\big)(s)\|_{L^p} ds \le CE_0^3(1+t)^{-\gamma-1}.$$

On the other hand, for the term $\partial_x I_{5,1}$, we apply (3.16) with $l=2$, $q=1$ and then with $l=1$, $q=1$, and combine them to obtain

$$\|\partial_x I_{5,1}\|_{L^p} \le C\int_0^{t/2} \|\partial_x^2 (G-G_0)(t-s) * (u^2)(s)\|_{L^p} ds \le CE_0^2(1+t)^{-\gamma-1},$$

where we have used the estimates $\|(u^2)(s)\|_{L^1} \le CE_0^2(1+t)^{-\frac{1}{2}}$ and $\|\partial_x^l(u^2)(s)\|_{L^p} \le CE_0^2(1+t)^{-\gamma-\frac{l+1}{2}}$ $(l=0,1)$. Also, applying (3.16) with $l=1$, $q=p$, we have

$$\|\partial_x I_{5,2}\|_{L^p} \le C\int_{t/2}^{t} \|\partial_x (G-G_0)(t-s) * \partial_x(u^2)(s)\|_{L^p} ds \le CE_0^2(1+t)^{-\gamma-1}\log(2+t).$$

Finally, we consider $\partial_x I_{6,1}$ and $\partial_x I_{6,2}$. For the term $\partial_x I_{6,1}$, we apply (7.4) with $l=2$, $q=1$ and then with $l=1$, $q=p$, and combine them to obtain

$$\begin{aligned}\|\partial_x I_{6,1}\|_{L^p} &\le C\int_0^{t/2} \|\partial_x^2 G_0(t-s) * (u^2-w^2)(s)\|_{L^p} ds \\ &\le C\int_0^{t/2} (t-s)^{-\frac{1}{2}}(1+t-s)^{-\gamma-\frac{1}{2}}\big(\|(u^2-w^2)(s)\|_{L^1} + \|\partial_x(u^2-w^2)(s)\|_{L^p}\big) ds.\end{aligned}$$

Here we observe that $\|u^2 - w^2\|_{L^1} \le \|u + w\|_{L^\infty}\|u - w\|_{L^1}$ and $\|\partial_x(u^2 - w^2)\|_{L^p} \le \|\partial_x(u^2)\|_{L^p} + \|\partial_x(w^2)\|_{L^p}$. Therefore, making use of (2.1), (2.7) and (7.7), we obtain

$$\|\partial_x I_{6,1}\|_{L^p} \le CE_0E_1\int_0^{t/2}(t-s)^{-\frac{1}{2}}(1+t-s)^{-\gamma-\frac{1}{2}}(1+s)^{-1+\varepsilon}ds \le C(\varepsilon)E_0E_1(1+t)^{-\gamma-1+\varepsilon}$$

for a constant $C(\varepsilon)$ depending ε. Also, applying (7.4) with $l = 1$, $q = p$, we have

$$\|\partial_x I_{6,2}\|_{L^p} \le C\int_{t/2}^t \|\partial_x G_0(t-s)*\partial_x(u^2-w^2)(s)\|_{L^p}ds \le C\int_{t/2}^t (t-s)^{-\frac{1}{2}}\|\partial_x(u^2-w^2)(s)\|_{L^p}ds.$$

Here we observe that

$$\|\partial_x(u^2 - w^2)\|_{L^p} \le \|u + w\|_{L^\infty}\|\partial_x(u-w)\|_{L^p} + \|\partial_x(u+w)\|_{L^p}\|u - w\|_{L^\infty}.$$

We know from (2.1) and the definition of $N(t)$ in (7.6) that the first term here is bounded by $CE_0N(t)s^{-\frac{1}{2}}(1+s)^{-\gamma-1+\varepsilon}$. Also, using (2.1) and (7.9), we can majorize the second term by $CE_0E_1(1+s)^{-\gamma-\frac{3}{2}+\varepsilon}$. Consequently, we obtain

$$\begin{aligned}\|\partial_x I_{6,2}\|_{L^p} &\le C(E_0N(t) + E_0E_1)\int_{t/2}^t (t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}}(1+s)^{-\gamma-1-\varepsilon}ds \\ &\le C(E_0N(t) + E_0E_1)(1+t)^{-\gamma-1-\varepsilon}.\end{aligned}$$

We can summarize all the above computations as

$$\|\partial_x(u-w)(t)\|_{L^p} \le CE_1t^{-\frac{1}{2}}(1+t)^{-\gamma-\frac{1}{2}} + C(\varepsilon)E_0E_1(1+t)^{-\gamma-1+\varepsilon} + CE_0N(t)(1+t)^{-\gamma-1+\varepsilon}. \tag{7.12}$$

This yields $N(t) \le CE_1 + C(\varepsilon)E_0E_1 + CE_0N(t)$, from which we can deduce the desired estimate $N(t) \le CE_1$ for suitably small E_0, say, $E_0 \le \delta_3(\varepsilon)$. Thus we obtain

$$\|\partial_x(u-w)(t)\|_{L^p} \le CE_1t^{-\frac{1}{2}}(1+t)^{-\gamma-\frac{1}{2}+\varepsilon}.$$

which together with (2.1) and (2.7) yields the desired estimate (7.11). This completes the proof of Theorem 2.3. □

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